Differentiable groupoids and their abstract Lie algebroids

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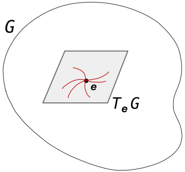
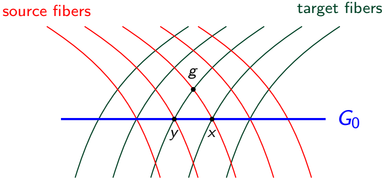
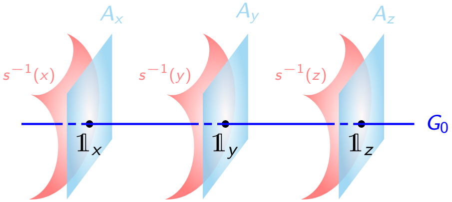
Infinitesimal counterparts of global structures

$$\text{Lie group(oid)s} \xrightarrow{\text{differentiation}} \text{Lie algebr(oid)s}$$







Questions

Question

Given a groupoid object G in a category $\mathcal{C}$, what are the **structures of** $\mathcal{C}$ *and the* **properties of** G *needed for its differentiation?*

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What are the infinitesimal objects in $\mathcal{C}$ that generalize Lie algebroids?

Motivation

Differentiation of diffeological group(oid)s.

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Definition (Rosický '84)

A **tangent structure** *on a category* $\mathcal{C}$ *is composed of an endofunctor* $T : \mathcal{C} \to \mathcal{C}$, *called an* **abstract tangent functor**, *together with morphisms*

(i) **bundle projection:** $\pi_X : TX \to X$, $\quad (u, u_1) \mapsto u$

(ii) **zero section:** $0_X : X \to TX$, $\quad u \mapsto (u, 0)$

(iii) **fiberwise addition:** $+_X : TX \times_X TX \to TX$, $\quad (u, u_1, v_1) \mapsto (u, u_1 + v_1)$

(iv) **vertical lift:** $\lambda_X : TX \to T^2X$, $\quad (u, u_1) \mapsto (u, 0, 0, u_1)$

(v) **symmetric str.:** $\tau_X : T^2X \to T^2X$, $\quad (u, u_1, u_2, u_{12}) \mapsto (u, u_2, u_1, u_{12})$

natural in $X \in \mathcal{C}$, *satisfying certain axioms.*

Suppose $\mathcal{C} = \mathcal{E}\mathrm{ucl}$ is the category of **Euclidean spaces** with objects open subsets $U \subset \mathbb{R}^n$ for some $n \geq 0$ and morphisms smooth maps $U \to V$. Let

$$T : \mathcal{E}\mathrm{ucl} \longrightarrow \mathcal{E}\mathrm{ucl}$$

be the usual tangent functor. Then,

$$TU = U \times \mathbb{R}^n$$
$$T^2U = U \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n$$
$$TU \times_U TU = U \times \mathbb{R}^n \times \mathbb{R}^n .$$

Definition (Rosický '84)

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(v) **symmetric structure:** $\tau_X : T^2X \to T^2X$,

natural in $X \in \mathcal{C}$, *satisfying certain axioms.*

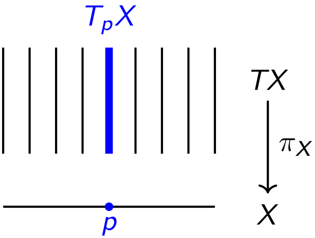
Examples 1

- The category of smooth manifolds.
- The category of elastic diffeological spaces (Blohmann '24).
- The category of smooth right G-bundles for G a Lie groupoid. For a G-bundle $r : E \to G_0$, it is given by $VE := TE \times_{TG_0}^{Tr, 0_{G_0}} G_0$.
- The category of abelian groups with $TA := A \times A$.

Axioms of a tangent structure

- **Fiber products**: For all $k \geq 1$, the pullbacks $\underbrace{TX \times_X \ldots \times_X TX}_{k\text{-times}}$ exist and are preserved by T.

- **Bundle of abelian groups**: The fiberwise addition $+_X$ and the zero section 0_X equip $\pi_X : TX \to X$ with the structure of a bundle of abelian groups (i.e. an abelian group object in $\mathcal{C} \downarrow X$).



Given a point $p : * \to X$, the fiber $T_pX := * \times_X TX$ is an abelian group.

Axioms of a tangent structure

- **Symmetric structure**: τ_X is a symmetric structure on TX and the diagram

$$\begin{array}{ccc} T^2X & \xrightarrow{\tau_X} & T^2X \\ & {\scriptstyle T\pi_X}\searrow \qquad \swarrow{\scriptstyle \pi_{TX}} & \\ & TX & \end{array}$$

  is a morphism of bundles of groups.

- **Vertical lift and that it is a kernel**: The following diagrams commute:

$$\begin{CD} TX @>\lambda_X>> T^2X \\ @V\pi_XVV @VV\pi_{TX}V \\ X @>>0_X> TX \end{CD} \qquad \begin{CD} TX @>\lambda_X>> T^2X \\ @V\lambda_XVV @VV\lambda_{TX}V \\ T^2X @>>T\lambda_X> T^3X \end{CD} \qquad \begin{CD} TX @>\lambda_X>> T^2X \\ @V\pi_XVV @VV(\pi_{TX},T\pi_X)V \\ X @>>(0_X,0_X)> TX \times_X TX \end{CD}$$

  The left diagram is a morphism of bundles of groups. The right diagram is a pullback.

- **Compatibility of vertical lift and symmetric structure**: The following diagrams commute:

$$\begin{array}{ccc} & TX & \\ & {\scriptstyle \lambda_X}\swarrow \qquad \searrow{\scriptstyle \lambda_X} & \\ T^2X & \xrightarrow[\tau_X]{} & T^2X \end{array} \qquad \begin{array}{ccccc} T^2X & \xrightarrow{T\lambda_X} & T^3X & \xrightarrow{\tau_{TX}} & T^3X \\ \downarrow{\scriptstyle \tau_X} & & & & \downarrow{\scriptstyle T\tau_X} \\ T^2X & & \xrightarrow[\lambda_{TX}]{\hspace{6em}} & & T^3X \end{array}$$

The Lie bracket

Tangent category $(\mathcal{C}, T) \longrightarrow$ **Lie bracket on vector fields**

Idea: Given two vector fields $v, w \in \Gamma(X, TX)$, define:

$$[v, w] \equiv Tw \circ v - \tau_X \circ Tv \circ w\,.$$

We obtain a vector field after applying the vertical lift λ.

Cartesian tangent structures

Definition

Let $(\mathcal{C}, T)$ be a tangent category.

- *It is called* **cartesian** *if the tangent functor preserves finite products.*
- *Let $R \in \mathcal{C}$ be a commutative ring object such that $TR \cong R \times R$. A* **scalar** R**-multiplication** *on $TX \to X$ is an R-module structure*

$$\kappa_X : R \times TX \longrightarrow TX\,,$$

together with some compatibility conditions.

In a cartesian tangent category $(\mathcal{C}, T)$, the tangent bundle $TX \to X$ is a bundle of R-modules (i.e. an R-module object in $\mathcal{C} \downarrow X$).

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Groupoid objects

Definition

A **groupoid object** *in a category* $\mathcal{C}$ *is a simplicial object* $G : \mathbf{\Delta}^{\mathrm{op}} \to \mathcal{C}$ *such that the limits*

$$G[\Lambda_i^n] := \lim_{\Delta^k \to \Lambda_i^n} G_k$$

exist in $\mathcal{C}$ *and the natural morphisms (horn projections)*

$$G_n \longrightarrow G[\Lambda_i^n]$$

are isomorphisms for all $n \geq 2$ *and all* $0 \leq i \leq n$*.*

Example 2

The object of composable pairs is given by $G_2 \cong G(\Lambda_1^2) \cong G_1 \times_{G_0}^{s,t} G_1$. Proceeding iteratively, we have

$$G_n \cong G_1 \times_{G_0} \ldots \times_{G_0} G_1 \,.$$

Differentiable groupoids

Definition (AB '24, Working definition)

A groupoid object G in a tangent category $(\mathcal{C}, T)$ is called **differentiable** *if the natural morphisms*

$$T(G_1 \times_{G_0} \ldots \times_{G_0} G_1) \longrightarrow TG_1 \times_{TG_0} \ldots \times_{TG_0} TG_1$$

are isomorphisms and if certain pullbacks exist.

In particular,

$$TG : \mathbf{\Delta}^{\mathrm{op}} \longrightarrow \mathcal{C},\ [n] \longmapsto (TG)_n := TG_n$$

is a groupoid object in $\mathcal{C}$.

Example 3

Lie groupoids are differentiable groupoid objects in the category of finite-dimensional smooth manifolds.

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Definition (AB '24)

An **abstract Lie algebroid** *in a cartesian tangent category* $(\mathcal{C}, T)$ *with scalar* R*-multiplication consists of:*

- *a bundle of* R*-modules* $A \to X$
- *a morphism* $\rho : A \to TX$ *of bundles of* R*-modules, called the* **anchor**
- *a Lie bracket on the abelian group* $\Gamma(X, A)$ *of sections,*

such that for all sections a*,* b *of* A *and all morphisms* $f : X \to R$ *in* $\mathcal{C}$*:*

$$[a, fb] = f[a, b] + \big((\rho \circ a) \cdot f\big) b \quad \textbf{Leibniz rule}$$
$$\rho \circ [a, b] = [\rho \circ a, \rho \circ b] \, .$$

- The abelian group $\Gamma(X, A)$ has a $\mathcal{C}(X, R)$-module structure:

$$fa : X \xrightarrow{(f,a)} R \times A \xrightarrow{\kappa} A \, .$$

- Vector fields on X (sections of TX) act on R-valued morphisms on X:

$$v \cdot f : X \xrightarrow{v} TX \xrightarrow{Tf} TR \cong R \times R \xrightarrow{\mathrm{pr}_2} R \, .$$

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Let G be a differentiable groupoid object in a tangent category $(\mathcal{C}, T)$ with a scalar R-multiplication.

Step 1: The s-vertical tangent bundle:

$$\begin{array}{ccc} \ker Ts := G_0 \times_{TG_0} TG_1 & \longrightarrow & TG_1 \\ \big\downarrow & \lrcorner & \big\downarrow Ts \\ G_0 & \xrightarrow[0_{G_0}]{} & TG_0 \end{array}$$

Proposition: $\pi'_{G_1} : \ker Ts \to TG_1 \to G_1$ is a bundle of R-modules.

Step 2: Restrict it to the identity bisection:

$$\begin{array}{ccc} A := G_0 \times_{G_1} \ker Ts & \longrightarrow & \ker Ts \\ \big\downarrow & \lrcorner & \big\downarrow \pi'_{G_1} \\ G_0 & \xrightarrow[1]{} & G_1 \end{array}$$

Proposition: $A \to G_0$ is a bundle of R-modules.

Step 3: Define the anchor ρ as the restriction of Tt to A:

$$\rho : A \to \ker Ts \to TG_1 \xrightarrow{Tt} TG_0 \, .$$

Proposition: The anchor is a morphism of bundles of R-modules.

Step 4: There is a right groupoid action $R : \ker Ts \times_{G_0} G_1 \to \ker Ts$, called the **right translation**. A vector field $v \in \Gamma(G_1, TG_1)$ is called **right invariant** if the diagrams

$$\begin{array}{ccc} G_1 & \xrightarrow{v} & TG_1 \\ & {\scriptstyle \widetilde{v}} \searrow \quad \nearrow & \\ & \ker Ts & \end{array} \qquad \begin{array}{ccc} G_1 \times_{G_0} G_1 & \xrightarrow{\widetilde{v} \times_{G_0} \mathrm{id}} & \ker Ts \times_{G_0} G_1 \\ {\scriptstyle m} \downarrow & & \downarrow {\scriptstyle R} \\ G_1 & \xrightarrow[\widetilde{v}]{} & \ker Ts \end{array}$$

commute for some morphism $\widetilde{v} : G_1 \to \ker Ts$.

Proposition (AB '24)

There is an isomorphism of $\mathcal{C}(G_0, R)$-modules

$$\mathfrak{X}(G_1)_{\text{r-inv}} \cong \Gamma(G_0, A)\,. \tag{1}$$

Proposition (AB '24)

Right invariant vector fields are closed under the Lie bracket of vector fields on G_1 (defined by the tangent structure on $\mathcal{C}$).

Theorem (AB '24)

Let G be a differentiable groupoid in $(\mathcal{C}, T)$. The bundle $A \to G_0$ with the anchor $\rho : A \to TG_0$ as defined before, and the Lie bracket on sections of A defined by (1) is an abstract Lie algebroid.

Upshot

Question

Given a groupoid object G in a category $\mathcal{C}$, what are the **structures of** $\mathcal{C}$ *and the* **properties of** G *needed for its differentiation?*

Answer:

- Tangent structure on $\mathcal{C}$
- Differentiability property on G (which makes T preserve the nerve of G and TG into a groupoid object)

Question

What are the infinitesimal objects in $\mathcal{C}$ that generalize Lie algebroids?

Answer: Abstract Lie algebroids

Future work

- Computing concrete examples of (elastic) diffeological groupoids
- What about higher differentiable groupoids/ higher abstract Lie algebroids in tangent categories?
- What about a Lie functor (morphisms of abstract Lie algebroids)

References

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- C. Blohmann. Elastic diffeological spaces. *Contemp. Math.*, 794:49-86, 2024.
- J. Rosický. Abstract tangent functors. *Diagrammes*, 12:JR1-JR11, 1984.

Thank you for your attention!